

STRONGLY HOMOTOPY LIE BIALGEBRAS AND LIE QUASI-BIALGEBRAS

OLGA KRAVCHENKO

This paper is dedicated to Jean-Louis Loday on the occasion of his 60th birthday with admiration and gratitude

Abstract

Structures of Lie algebras, Lie coalgebras, Lie bialgebras and Lie quasibialgebras are presented as solutions of Maurer-Cartan equations on corresponding governing differential graded Lie algebras using the big bracket construction of Kosmann-Schwarzbach. This approach provides a definition of an L_∞ -(quasi)bialgebra (strongly homotopy Lie (quasi)bialgebra). We recover an L_∞ -algebra structure as a particular case of our construction. The formal geometry interpretation leads to a definition of an L_∞ (quasi)bialgebra structure on V as a differential operator Q on V , self-commuting with respect to the big bracket. Finally, we establish an L_∞ -version of a Manin (quasi) triple and get a correspondence theorem with L_∞ -(quasi) bialgebras.

1. INTRODUCTION.

Algebraic structures are often defined as certain maps which must satisfy quadratic relations. One of the examples is a Lie algebra structure: a Lie bracket satisfies the Jacobi identity (indeed the Jacobi identity is a quadratic relation since the bracket appears twice in each summand). Other examples include an associative multiplication (the associativity condition is quadratic), L_∞ and A_∞ algebras (also called strongly homotopy Lie and strongly homotopy associative algebras) and many others.

The subject of this article is a description of Lie (quasi) bialgebras and their L_∞ -versions. The main philosophy is that the axioms of Lie (quasi)bialgebras (and their L_∞ -versions) could be written in the form of a quadratic relation on a certain governing differential graded Lie algebra. We find the governing differential graded Lie algebras for Lie bialgebra and Lie quasi-bialgebra structures using Kosmann-Schwarzbach's big-bracket construction [12]. The L_∞ brackets are obtained by using the (higher) derived brackets [15, 30, 1].

The quadratic relation on the structure can be expressed in the form of a Maurer-Cartan equation.

Classically, solutions of the Maurer-Cartan equation are considered only from the first graded component of the governing differential graded Lie algebra and they give the original algebraic structure. However solutions of the Maurer-Cartan equation in the whole governing differential graded Lie algebra provide a strongly homotopy version of the original one.

Let us give here the definitions of Lie bialgebras, Lie quasibialgebras and Manin triples and pairs to start with.

A good reference on Lie (quasi)bialgebras is the book by Etingof and Schiffmann ([7], pages 32–34 and 150–152).

Definition 1. *A Lie bialgebra structure on a vector space V is the following data:*

- a:** *a Lie bracket, $\{\cdot, \cdot\} : V \wedge V \rightarrow V$;*
- b:** *a Lie cobracket, that is an element $\delta : V \rightarrow V \wedge V$, satisfying the coJacobi identity:*

$$Alt(\delta \otimes 1)\delta(x) = 0$$

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c: a compatibility condition between $\{\cdot, \cdot\}$ and δ , meaning that δ is a 1-cocycle: $\delta(\{x, y\}) = \{\delta(x), 1 \otimes y + y \otimes 1\} + \{1 \otimes x + x \otimes 1, \delta(y)\}$,

where for x, y, z elements of V we denote $Alt(x \otimes y \otimes z) = x \otimes y \otimes z + y \otimes z \otimes x + z \otimes x \otimes y$.

To distinguish a Lie bracket on the governing differential graded Lie algebra from a Lie bracket in a Lie (quasi)bialgebra structure we denote the former one by $[\cdot, \cdot]$ and the latter one by $\{\cdot, \cdot\}$ throughout the article.

Definition 2. A Lie quasibialgebra structure on a vector space V is the following data:

a: a Lie bracket $\{\cdot, \cdot\}$;

b: an element $\delta \in Hom(V, V \wedge V)$, and an element $\phi \in V \wedge V \wedge V$ satisfying a modified coJacobi identity:

$$\frac{1}{2} Alt(\delta \otimes 1) \delta(x) = \{x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x, \phi\}$$

$$\text{and } (\delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta)(\phi) = 0.$$

c: δ is a 1-cocycle with respect to the bracket $\{\cdot, \cdot\}$.

The main difficulty in defining strongly homotopy versions of Lie bialgebras and Lie quasibialgebras in the generalized Maurer-Cartan approach is in finding the corresponding governing Lie algebra. Let $\bigwedge V$ be the vector space be the exterior power vector space. Then the Lie bracket and the Lie cobracket on V could be considered as elements of the space of homomorphisms $Hom(\bigwedge V, \bigwedge V)$. However, the Lie algebra structure on this space $Hom(\bigwedge V, \bigwedge V)$ is not obvious. Actually, its associative analogue, $Hom(\otimes V, \otimes V)$, does not have such a structure at all (which makes the Lie bialgebra quantization problem so intriguing).

We define the correct Lie algebra structure on $Hom(\bigwedge V, \bigwedge V)$ by considering an isomorphism of this space to $\bigwedge V^* \otimes \bigwedge V$. The Lie algebra structure on the latter is given by the Kosmann-Schwarzbach big bracket ([12]), which is also just the Batalin-Vilkovisky odd Poisson bracket on the space $V^* \otimes V$ with a shifted degree ([4, 9]).

The isomorphism from $\bigwedge V^* \otimes \bigwedge V$ to $Hom(\bigwedge V, \bigwedge V)$ is obtained by using the iterated adjoint Lie action. In the part “ L_∞ -structures” we first interpret the strongly homotopy Lie algebra structure in terms of the iterated adjoint action and then generalize the construction to the case of Lie (quasi)bialgebras. This might look as a complicated way to define an L_∞ -structure while a definition by coderivations of $\bigwedge^k V$ is more conventional. However, a Lie (quasi)-bialgebra structure is not given by a coderivation. The higher derived brackets introduced by Th. Voronov ([30]) and also studied by Akman-Ionescu ([1]) provide the necessary tool.

Finite dimensional Lie bialgebras are in one-to-one correspondence with Manin triples:

Definition 3. A Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a triple of finite dimensional Lie algebras, where $\mathfrak{g}_+ \oplus \mathfrak{g}_- = \mathfrak{g}$ as a vector space and \mathfrak{g} is equipped with a nondegenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ such that \mathfrak{g}_+ and \mathfrak{g}_- are Lagrangian subalgebras (that is maximal isotropic subspaces which are Lie subalgebras).

Lie quasi-bialgebras turn out to be described by the notion of a Manin pair.

Definition 4. A Manin pair is a pair $(\mathfrak{g}, \mathfrak{g}_+)$ where \mathfrak{g} is a finite dimensional Lie algebra equipped with a non-degenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ and \mathfrak{g}_+ is a Lagrangian subalgebra.

A Manin quasi-triple (also called a marked pair) is a pair $(\mathfrak{g}, \mathfrak{g}_+)$ with a chosen Lagrangian complement of \mathfrak{g}_+ .

The main theorem here is by Drinfeld [6] which states that Manin quasi-triples are in one-to-one correspondence with Lie quasi-bialgebras.

We will formulate and prove an L_∞ (in other words strongly homotopy) version of the Lie (quasi)bialgebras - Manin (quasi)triples correspondence.

It should be mentioned that an operad [20] or rather properad (or PROP) approach is not used in this paper. However, the definition of an L_∞ -bialgebra coming from a minimal resolution of a Lie bialgebra PROP coincides with ours as shown explicitly in the work of Sergei Merkulov [23, Corollary 5.2] and also could be derived from works [10] on dioperads and [21, 28] on Koszul PROPs, the Lie bialgebra PROP being one of them.

To avoid confusion, it should also be mentioned that if one wants to consider an associative bialgebra there is no similar construction of a governing Lie algebra since the bialgebra PROP is not Koszul. Our method works only for Lie (quasi)bialgebras.

We use the Koszul sign convention: in a graded algebra whenever there is a change of places of two symbols there should be a corresponding sign. Throughout this paper, the summation convention is understood: indices α, β, \dots once as superscript and once as subscript in a formula are to be summed over.

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2. KOSMANN-SCHWARZBACH'S BIG BRACKET

Treating exterior powers of a sum of a vector space with its dual as a super-Poisson algebra was pioneered in the works of Batalin-Fradkin-Vilkovisky [9, 4] and later of Stasheff [27] and Kostant-Sternberg [16]. In 1991 Yvette Kosmann-Schwarzbach published an article [12] where the term *big bracket* was introduced in order to describe proto-bialgebras (a notion generalizing Lie bialgebras). This big bracket defines, in particular, the Lie structure of the governing dgLie algebra of Lie bialgebras (and, in fact, Lie algebras, Lie coalgebras and Lie quasibialgebras).

Here is the construction from [12] in a \mathbb{Z} -graded context. On a direct sum of a vector space and its dual there is a symmetric bracket pairing the space with its dual. There is a way to see this symmetric bracket as a graded odd Lie bracket on a graded space.

On any \mathbb{Z} -graded space $X = \sum X^i$ there is an operation called de-suspension, mapping X to the space $X[1]$ so that $X[1]_i = X_{i+1}$. This shift of the degree allows in particular to see symmetric powers of the space X as exterior powers of the shifted space $X[1]$ (and vice-versa). There is the following natural identification of symmetric and exterior powers of X :

$$(1) \quad \text{Sym}^n(X) \simeq \left(\bigwedge^n(X[1]) \right) [-n].$$

Thus a symmetric form acting from $\text{Sym}^2(X)$ to $\mathbb{K} = \text{Sym}^0(X)$ becomes a bracket

$$\left(\bigwedge^2(X[1]) \right) [-2] \rightarrow \bigwedge^0(X[1]) = \mathbb{K}.$$

This bracket could also be viewed as a map: $\bigwedge^2(X[1]) \rightarrow \left(\bigwedge^0(X[1])\right)[2] = \mathbb{K}[2]$. If X is ungraded we get elements of degree -1 in $X[1]$ and the $\mathbb{K}[2]$ is in degree -2 . Thus the bracket has degree 0, since two elements of degree -1 are sent by the bracket to $\mathbb{K}[2]$ of degree $(-1) + (-1) = -2$.

Let \tilde{V} be a finite dimensional \mathbb{K} -vector space ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}), and \tilde{V}^* its dual. A non-graded space $\tilde{V}^* \oplus \tilde{V}$ could be considered as a graded one by assigning degree 0 to each of its elements. Consider $V^* \oplus V = (\tilde{V}^* \oplus \tilde{V})[1]$, this means that “points” of $V^* \oplus V$ are in degree -1 . Algebraic functions on $\tilde{V} \oplus \tilde{V}^*$ form the space

$$(2) \quad B = \bigoplus_{j \geq -2} B^j, \quad \text{where } B^j = \bigoplus_{p+q=j} \left(\bigwedge^{p+1} V^* \otimes \bigwedge^{q+1} V \right), \quad j \geq -1, \quad \text{and } B^{-2} = \mathbb{K}.$$

From this point of view B is an algebra of exterior powers of the odd space $V^* \oplus V$.

Grading in B is given by the sum $p+q$. That is, B^j consists of terms with $j = p+q$, for $p, q \geq -1$, so that the first few terms are as follows:

$$\begin{aligned} B^{-2} &= k, \\ B^{-1} &= V \oplus V^*, \\ B^0 &= V \wedge V \oplus V^* \otimes V \oplus V^* \wedge V^*, \\ B^1 &= V \wedge V \wedge V \oplus V^* \otimes V \wedge V \oplus V^* \wedge V^* \otimes V \oplus V^* \wedge V^* \wedge V^*. \end{aligned}$$

The space B is bigraded as follows:

		...				
2		$V^* \wedge V^* \wedge V^*$...			
1		$V^* \wedge V^*$	$V^* \wedge V^* \otimes V$...		
0		V^*	$V^* \otimes V$	$V^* \otimes V \wedge V$...	
-1		\mathbb{K}	V	$V \wedge V$	$V \wedge V \wedge V$...
$p \quad / \quad q$		-1	0	1	2	

Let $\langle \cdot, \cdot \rangle$ be the natural pairing of \tilde{V} and \tilde{V}^* . We extend it to a symmetric form on $\tilde{V} \oplus \tilde{V}^*$ as follows: for $x, y \in \tilde{V}$ and $v, w \in \tilde{V}^*$:

$$\langle x + v, y + w \rangle = \langle x, y \rangle + \langle v, w \rangle.$$

This symmetric form on $\tilde{V} \oplus \tilde{V}^*$ could be considered as an antisymmetric form on the de-suspended space $V \oplus V^*$. Moreover, on B it gives a Lie algebra structure by the following

Definition 5. The big bracket is the graded Lie algebra structure on algebraic functions on $V \oplus V^*$ defined as follows.

- For $u, v \in B^{-2} \oplus B^{-1} = k \oplus V \oplus V^*$:

$$[u, v] = \begin{cases} \langle u, v \rangle & \text{if } u \in V \oplus V^* \text{ and } v \in V \oplus V^* \\ 0 & \text{if } u \in k \text{ or } v \in k \end{cases}$$

- The bracket on other terms is defined by linearity and the graded Leibniz rule: for $u \in B^k$, $v \in B^l$, $w \in B^m$

$$[u, v \wedge w] = [u, v] \wedge w + (-1)^{kl} v \wedge [u, w]$$

- Remark 6.** 1) $[u, v] = -(-1)^{kl}[v, u]$; $u \in B^k$, $v \in B^l$, that is the big bracket is skew-symmetric in the graded sense;
 2) $[\cdot, \cdot] : B^i \wedge B^j \rightarrow B^{i+j}$, in other words: the bracket is of degree zero;
 3) in particular,

$$[\cdot, \cdot] : (\bigwedge^k V^* \otimes \bigwedge^l V) \wedge (\bigwedge^m V^* \otimes \bigwedge^n V) \longrightarrow \bigwedge^{k+m-1} V^* \otimes \bigwedge^{l+n-1} V;$$

- 4) B^0 is a Lie subalgebra of B ;
 5) B^{-2} is the center of $(B, [\cdot, \cdot])$.

3. GOVERNING GRADED LIE ALGEBRAS AND MAURER-CARTAN EQUATIONS

The graded Lie algebra B has several graded Lie subalgebras. Verification that they are indeed Lie subalgebras is easy using Remark 6 (3). Some of these Lie subalgebras give well known structures as was described in [12]. We put them in the following table containing the $-1, 0$ and 1 graded components of the governing Lie algebras:

\mathfrak{g} - Lie subalgebra of B	\mathfrak{g}^{-1}	\mathfrak{g}^0	\mathfrak{g}^1	Solutions of Maurer-Cartan equation
$\mathcal{L} :$ column $q = 0$	$u \in V$	$f \in V^* \otimes V$	$l \in V^* \wedge V^* \otimes V$	$[l, l] = 0$ Lie algebra structure on V
$\mathcal{C} :$ row $p = 0$	$u^* \in V^*$	$f \in V^* \otimes V$	$c \in V^* \otimes V \wedge V$	$[c, c] = 0$ Lie coalgebra structure on V
$\mathcal{B} :$ $p, q \neq -1$	—	$f \in V^* \otimes V$	$c \in V^* \otimes V \wedge V$ $l \in V^* \wedge V^* \otimes V$	$[c + l, c + l] = 0$ Lie bialgebra structure on V
$\mathcal{QB} :$ $q \neq -1$	$u \in V$	$f \in V^* \otimes V$ $g \in V \wedge V$	$c \in V^* \otimes V \wedge V$ $l \in V^* \wedge V^* \otimes V$ $\phi \in V \wedge V \wedge V$	$[c + l + \phi, c + l + \phi] = 0$ Lie quasi-bialgebra structure on V

Let $\bigwedge V$ denote the sum $k \oplus V \oplus \bigwedge^2 V \oplus \bigwedge^3 V \oplus \dots$. Then $\mathcal{C} = V^* \otimes \bigwedge V \simeq \text{Hom}(V, \bigwedge V)$, and $\mathcal{L} = \bigwedge V^* \otimes V \simeq \text{Hom}(\bigwedge V, V)$. In other words,

Proposition 7. *On a vector space V*

Lie algebras : $\mathcal{L} = \bigoplus_{k \geq 0} \bigwedge^k V^* \otimes V$ is the governing graded Lie algebra of Lie algebra structures on V . In particular, an element of degree 1, l such that $[l, l] = 0$ defines a Lie algebra structure $\{\cdot, \cdot\} \in \text{Hom}(V \wedge V, V)$ as follows: $\{x, y\} = [[l, x], y]$.

Lie coalgebras: $\mathcal{C} = V^* \otimes (\bigoplus_{l \geq 0} \bigwedge^l V)$ is the governing graded Lie algebra of Lie coalgebra structures on V . In particular, an element c of degree 1 such that $[c, c] = 0$ defines a Lie coalgebra structure $\delta \in \text{Hom}(V, V \wedge V)$ as follows: $\delta(x) = [c, x]$.

Lie bialgebras: $\mathcal{B} = \bigoplus_{k \geq 1, l \geq 1} \bigwedge^k V^* \otimes \bigwedge^l V$ is the governing graded Lie algebra of Lie bialgebra structures on V . In particular, elements of degree 1, $\theta \in V^* \otimes V \wedge V$ and $l \in V^* \wedge V^* \otimes V$

such that $[c + l, c + l] = 0$ define a Lie bialgebra structure with the cobracket $\delta(x) = [c, x]$ and the bracket $\{x, y\} = [[l, x], y]$.

Lie quasi-bialgebras: $\mathcal{QB} = \bigoplus_{k \geq 0, l \geq 0} \bigwedge^k V^* \otimes \bigwedge^l V$ is the governing graded Lie algebra of Lie quasibialgebra structures on V . In particular, elements of degree 1 :

$$c \in V^* \otimes V \wedge V, l \in V^* \wedge V^* \otimes V \text{ and } \phi \in V \wedge V \wedge V \text{ such that}$$

$$(3) \quad [c + l + \phi, c + l + \phi] = 0$$

define a Lie quasibialgebra structure with the 1-cocycle $\delta(x) = [c, x]$, the Lie bracket $\{x, y\} = [[l, x], y]$ and the 3-tensor ϕ .

Proof. Under the identification $\text{Hom}(\bigwedge^m V, \bigwedge^n V) \simeq \bigwedge^m V^* \otimes \bigwedge^n V^*$ the condition $[l, l] = 0$ is exactly the Jacobi identity.

The equation $[c, c] = 0$ is exactly the co-Jacobi identity on the corresponding $\delta \in \text{Hom}(V, V \wedge V)$.

The commutator $[l + c, l + c]$ lies in $(\bigwedge^3 V^* \otimes V) \oplus (V^* \otimes \bigwedge^3 V) \oplus (\bigwedge^2 V^* \otimes \bigwedge^2 V) \subset B$. Then $l + c$ defines a Lie bialgebra structure if its commutator with itself is 0, that is all three components of it in B^2 must be equal to 0. This leads to the three axioms of a Lie bialgebra:

- a: Jacobi identity follows from $[l, l] = 0$, for the derived bracket given by $\{x, y\} = [[l, x], y]$.
- b: Co-Jacobi identity follows from $[c, c] = 0$.
- c: The cocycle condition translates as: $[l, c] = 0$.

In fact, a presentation of a Lie bialgebra structure as a square zero element in $\bigwedge(V^* \oplus V)$ appeared first in [18] (see also [25] for the idea of a derived bracket involved).

In the same manner the equation 3 gives independent equations: $[l, l] = 0$, $[c, c] + 2[l, \phi] = 0$, $[c, \phi] = 0$ and $[c, l] = 0$, which give the axioms of a Lie quasibialgebra.

One could look for complete proofs in [12]. □

In all the cases of Proposition 7 we get a graded Lie algebra with a differential d given by the adjoint action of an auto-commuting element in the first degree. Hence $\mathcal{C}, \mathcal{L}, \mathcal{B}$ and \mathcal{QB} become dgLie algebras. Here is a general fact:

Proposition 8. [13] *A differential graded Lie algebra $(\mathfrak{g}, [\cdot, \cdot], D)$ gives rise to a new bracket of degree +1, called a derived bracket with respect to D : for $a, b \in \mathfrak{g}$, $\{a, b\} = [Da, b]$. This is a Loday-Leibniz algebra bracket in the sense of [19]. If W is an Abelian subalgebra of \mathfrak{g} , such that $[DW, W]$ is in W , then $\{\cdot, \cdot\}$ is a Lie bracket on W . Moreover, $D : (W, \{\cdot, \cdot\}) \rightarrow (\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra morphism.*

In our case, the algebra \mathfrak{g} is one of $\mathcal{C}, \mathcal{L}, \mathcal{B}$ and \mathcal{QB} , while the subalgebra $\text{Lie } W$ in all cases is the same $B^{-1} = V^* \oplus V$. The derived bracket with respect to corresponding differentials on $\mathcal{C}, \mathcal{L}, \mathcal{B}$ will define the Lie bracket on $V^* \oplus V$, leading to Manin triples (we could see a Lie structure on V as a particular case of a Manin triple with a zero cobracket, and a Lie coalgebra structure as a Manin triple with zero Lie bracket). In the same way, we get a marked Manin pair from \mathcal{QB} .

However, we need a certain refinement of Proposition 8, since the Lie subalgebra B^{-1} is not Abelian.

Proposition 9. *The differential graded Lie algebra $(B, [\cdot, \cdot], d)$, gives rise to a derived bracket of degree 1: $\{a, b\} = [da, b]$.*

This derived bracket restricted to B^{-1} is a Lie bracket and $d : (B^{-1}, \{\cdot, \cdot\}) \rightarrow (B^0, [\cdot, \cdot])$ is a Lie algebra morphism.

Proof. Notice that $\{B^{-1}, B^{-1}\} \subset B^{-1}$. The subspace $B^{-1} = V^* \oplus V$ is not an Abelian Lie subalgebra of $(B, [\cdot, \cdot])$, and we cannot use Proposition 8. However, since the bracket $[\cdot, \cdot]$ on B^{-1} takes values

in the center of B (namely, $B^{-2} = k$), the Jacobi identity on B gives us the skew-symmetry of the derived bracket $\{\cdot, \cdot\}$. Hence the derived bracket defines a Lie algebra structure (not just Loday's) on $V^* \oplus V$.

The Lie algebra morphism part is immediate: $d\{a, b\} = d[da, b] = [da, db]$. \square

Finally, to make explicit the connection to Manin pairs and triples we have the following

Proposition 10. *Let $l \in \bigwedge^2 V^* \otimes V$, $c \in V^* \otimes \bigwedge^2 V$, $\phi \in \bigwedge^3 V$. Then $(V \oplus V^*, \{a, b\} = [da, b])$, for*

- $d = ad_{l+c}$, with a condition $[l + c, l + c] = 0$ defines a Manin triple $(V \oplus V^*, V, V^*)$.
- $d = ad_{l+c+\phi}$, satisfying $[l + c + \phi, l + c + \phi] = 0$ defines a Manin pair $(V \oplus V^*, V)$.

4. L_∞ STRUCTURES

In the previous section we have seen that a Lie algebra structure on a space V is obtained as a derived bracket defined on the graded Lie algebra \mathcal{L} with a differential given by an adjoint action of an element from $V^* \wedge V^* \otimes V \subset B^1$ whose bracket with itself is 0.

If we take an element from $\mathcal{L} = \bigoplus_{k=0}^\infty \bigwedge^k V^* \otimes V$ (not just from $V^* \wedge V^* \otimes V$) whose bracket with itself is 0 we could define an L_∞ structure on V using higher derived brackets ([30], [1]). An autocommuting element from \mathcal{B} (or \mathcal{QB}) defines an L_∞ bialgebra (or L_∞ quasi-bialgebra) structure using iterated adjoint action via derived brackets. Here we develop this theory.

A certain subtlety is in defining what an element of degree 1 would mean in this context. For that we need to introduce a new notion of degree.

4.1. Degree. A starting point of any homotopy construction is a graded vector space $(W = \bigoplus_a W^a, \delta)$, with a differential of degree 1, in other words a complex

$$(4) \quad \cdots \xrightarrow{\delta} W^a \xrightarrow{\delta} W^{a+1} \xrightarrow{\delta} \cdots$$

Let the space V be graded. Then the space $B = \bigoplus_{p,q \geq -1} (\bigwedge^{p+1} V^* \otimes \bigwedge^{q+1} V)$, as well as the space of maps $\bigwedge V \rightarrow \bigwedge V$, inherit the grading from V in a consistent way. We define a grading and a differential on these spaces as follows.

Lemma 11. *Let $(V = \bigoplus_a V^a, \delta)$ be a differential graded space over an ungraded field \mathbb{K} . Then on the dual space V^* there is a corresponding grading with the opposite sign and a differential d of degree 1, $d : V^*_{-(a+1)} \rightarrow V^*_{-a}$:*

$$(5) \quad \cdots \xleftarrow{d} V^*_{-a} \xleftarrow{d} V^*_{-(a+1)} \xleftarrow{d} \cdots$$

Proof. If u is in V^a and $v = \delta u \in V^{a+1}$ then if $v^* \in V^*$ is a dual of v let us define: $u^* = dv^*$, where d acts as an adjoint operator on the dual space. This element dv^* is then dual to u :

$$(6) \quad 1 = [v^*, v] = [v^*, \delta u] = [dv^*, u] = [u^*, u].$$

From the duality of graded spaces V and V^* we get that d is of degree 1 on V^* . \square

This way $(V^* \oplus V, d + \delta)$ becomes a complex. We could extend the action of $d + \delta$ on $\bigwedge^p V^* \otimes \bigwedge^q V$ for any positive p, q by the Leibniz rule, so that the whole B becomes a complex.

Let us define a new grading on the complex B , taking into account the grading on V :

Definition 12. *Given a graded vector space $V = \bigoplus_a V^a$, consider elements from $\bigwedge^p V^* \otimes \bigwedge^q V$. The degree of elements from V^* is defined according to Lemma 11, and $\bigwedge^0 V = \bigwedge^0 V^* = \mathbb{K}$ is of degree 0.*

Let $w = x_1^ x_2^* \dots x_p^* y^1 y^2 \dots y^q \in \bigwedge^p V^* \otimes \bigwedge^q V$, such that x_i^*, y^j are homogeneous, that is belonging to just one graded component of $V^* \oplus V$, the degree is denoted by \sim . Then the internal degree of w is*

the following sum of the degrees $\mathbf{id}(w) = \sum \widetilde{x}_i^* + \sum \widetilde{y}^i$. The external degree comes from the grading of B : $\mathbf{ed}(w) = p + q - 2$.

The total degree of w then is $\mathbf{td}(w) = \mathbf{id}(w) + \mathbf{ed}(w) = \sum \widetilde{x}_i^* + \sum \widetilde{y}^i + (p + q - 2)$.

A map $\tau_{pq} : \bigwedge^p V \rightarrow \bigwedge^q V$ sending $v^1 v^2 \dots v^p$ to $u^1 u^2 \dots u^q$ has the degree n if $(\sum \widetilde{u}_i) - (\sum \widetilde{v}_j) + (p + q - 2) = n$.

Remark 13. Let v_i^* be defined by the following property: $[v_i^*, v^i] = 1$. Then a map τ_{pq} has degree n if and only if $t_{pq} = v_1^* v_2^* \dots v_p^* u^1 u^2 \dots u^q \in \bigwedge^p V^* \otimes \bigwedge^q V$ has total degree n .

Let us define the solutions of the Maurer-Cartan equation of total degree 1 in B . The correction of $(p + q - 2)$ takes into account that elements entering the Maurer-Cartan equation are no longer necessarily from B^1 (that is when $p + q = 3$) but from B^{p+q-2} for any p and q .

4.2. L_∞ algebra structure from the iterated adjoint action. Let us recall the definition of L_∞ -algebras.

Definition 14. Consider a map $\lambda_k \in \text{Hom}(\bigwedge^k V, V)$. This map λ_k acts on $\bigwedge^n V$ for any $n \in \mathbb{N}$ by coderivation of the unshuffle coproduct on the algebra of exterior powers of V :

$$(7) \quad \lambda_k(v_1 \wedge \dots \wedge v_n) = \begin{cases} 0, & \text{if } k > n \\ \sum_{\sigma \in Sh_n^k} (-1)^{sgn\sigma} \lambda_k(v_{\sigma(1)} \wedge \dots \wedge v_{\sigma(k)}) \wedge v_{\sigma(k+1)} \wedge \dots \wedge v_{\sigma(n)}, & \text{otherwise.} \end{cases}$$

The set Sh_n^k is the set of all k -unshuffles in the permutation group of n elements, that is all permutations such that $\sigma(1) < \sigma(2) < \dots < \sigma(k)$ and $\sigma(k+1) < \dots < \sigma(n)$.

Definition 15. An L_∞ -structure on a graded space V is a set of maps of total degree 1,

$$\lambda_k : \bigwedge^k V \rightarrow V[2 - k], \quad k \in \mathbb{N}$$

such that the following generalized form of the Jacobi identity is satisfied for any $n \geq 2$:

$$(8) \quad \sum_{k=1}^{n-1} [\lambda_k, \lambda_{n-k}] = 0.$$

The bracket $[\cdot, \cdot]$ here is the commutator of λ_k and λ_{n-k} considered as operators on $\bigwedge V$. These equations are higher Jacobi identities and can be summarized in one equation:

$$(9) \quad \lambda^2 = 0, \quad \text{where } \lambda = \sum_{k \geq 1} \lambda_k.$$

For a finite dimensional space V there is the following isomorphism $\text{Hom}(\bigwedge^k V, \bigwedge^l V) \simeq \bigwedge^k V^* \otimes \bigwedge^l V$. Any operator $\tau_{kl} \in \text{Hom}(\bigwedge^k V, \bigwedge^l V)$ is represented by $t_{kl} \in \bigwedge^k V^* \otimes \bigwedge^l V$ and a passage from t_{kl} to τ_{kl} could be made explicit by introducing the iterated adjoint action.

Definition 16. Consider an element $t_{kl} \in \bigwedge^k V^* \otimes \bigwedge^l V$. The iterated adjoint action of t_{kl} on $\bigwedge^n V$ is defined as follows. For $n = k$ it defines a higher derived bracket as in [1, 29]:

$$(10) \quad \tau_{kl}(v_1 \wedge \dots \wedge v_k) = \mathbf{iad}_{t_{kl}}(v_1 \wedge \dots \wedge v_k) = [[\dots [t_{kl}, v_1], v_2] \dots], v_k],$$

thus defining a map $\bigwedge^k V^* \otimes \bigwedge^l V \rightarrow \text{Hom}(\bigwedge^k V, \bigwedge^l V) : t_{kl} \mapsto \tau_{kl}$. More generally, we define an action of t_{kl} on any exterior power of V , with values in $\bigwedge V^* \otimes \bigwedge V$ as follows

$$(11) \quad \begin{aligned} & \mathbf{iad}_{t_{kl}}(v_1 \wedge \dots \wedge v_n) \\ &= \begin{cases} [[\dots [t_{kl}, v_1], v_2] \dots], v_n], & \text{if } k \geq n \\ \sum_{\sigma \in Sh_n^k} (-1)^{sgn\sigma} \underbrace{[\dots [t_{kl}, v_{\sigma(1)}, v_{\sigma(2)}] \dots]}_{k \text{ times}}, v_{\sigma(k)} \wedge v_{\sigma(k+1)} \wedge \dots \wedge v_{\sigma(n)}, & \text{otherwise.} \end{cases} \end{aligned}$$

The set of permutations Sh_n^k is as in Definition 14. For the case $n = k$ one should keep in mind that $\bigwedge^0 V^* = \mathbb{K}$.

It is shown already in [30] that an L_∞ -algebra can be obtained as a particular case of this iterated adjoint action.

We now state two theorems which are L_∞ analogues of results of Proposition 8.

Theorem 17. (see [30]) *Consider an element $L \in \mathcal{L}$, $L = \sum_{k=1}^\infty l_k$, $l_k \in \bigwedge^k V^* \otimes V$ of total degree 1, such that $[L, L] = 0$. The set of maps $\lambda_k = \mathbf{iad}_{l_k} \in \text{Hom}(\bigwedge^k V, V)$ form an L_∞ -algebra.*

In other words, $[L, L] = 0$ implies that $\lambda^2 = 0$ where $\lambda = \mathbf{iad}_L$.

The proof is a direct albeit tedious computation reducing the condition on the maps λ_k to $[L, L] = 0$ using the fact that $\bigwedge V$ is an Abelian Lie subalgebra of \mathcal{L} and also that $[L, v] \in \bigwedge V^*$ for any $v \in \bigwedge V^*$.

We also use the following statement analogous to the one about the Lie algebra morphism in Proposition 8.

Definition 18. *An L_∞ -morphism from an L_∞ -algebra $(V, \sum \lambda_k)$ to a Lie algebra $(W, [,])$ is a sequence of maps $\phi_l : \bigwedge^l V \rightarrow W$ of total degree 0 such that we have the following equality, for all $n \geq 2$,*

$$\sum_{k+l=n} \phi_l \lambda_k = \sum_{k+l=n+1} [\phi_k, \phi_l].$$

Theorem 19. *Consider a graded space V with an L_∞ -structure given by the iterated adjoint action of $L = \sum l_k$, for $l_k \in \bigwedge^k V^* \otimes V$, and $[L, L] = 0$. Consider the graded Lie algebra structure on $V^* \otimes V$ with the bracket $[,]$ given by the natural pairing.*

Then maps

$$\mathbf{iad}_{l_{k+1}} : \bigwedge^k V \rightarrow V^* \otimes V$$

define an L_∞ morphism $(V, \sum \mathbf{iad}_{l_k}) \rightarrow (V^ \otimes V, [,])$.*

Checking this proposition amounts just to writing the higher Jacobi identities (8) on V .

Remark 20. *In [3, Corollary 2] it is noticed that an odd self-commuting element from B defines an L_∞ structure on $\bigwedge V$ (as well as on $\bigwedge V^*$) by the iterated adjoint action.*

4.3. L_∞ -coalgebras.

Definition 21. *An L_∞ coalgebra structure on V is a sequence of maps*

$$\delta_p : V \rightarrow \bigwedge^p V \quad [2-p], \quad l \geq 1$$

of total degree 1 such that

$$\sum_{p \geq 1, q \geq 1} \delta_p \delta_q = 0.$$

In other words, $\delta^2 = 0$, where $\delta = \sum_{p \geq 1} \delta_p$.

Consider the Lie algebra $\mathcal{C} = \bigoplus_{q \geq 0} V^* \otimes \bigwedge^{q+1} V$, governing the Lie coalgebra structure on V . The adjoint action of $c_p \in V^* \otimes \bigwedge^p V$ on $\bigwedge^n V$ is a map from $\bigwedge^n V$ to $\bigwedge^{n+p-1} V$:

$$ad_{c_p}(v_1 \wedge \cdots \wedge v_n) = \sum_{i=1}^n (-1)^{sgn} [c_p, v_i] \wedge v_1 \wedge \cdots \check{v}_i \cdots \wedge v_n.$$

Theorem 22. *Let $C = \sum_{p=1}^\infty c_p$, for $c_p \in V^* \otimes \bigwedge^p V$ be such that $[C, C] = 0$. Then the adjoint action of C defines an L_∞ -coalgebra structure on V .*

Moreover, the iterated adjoint action of C on $\bigwedge V^$ defines an L_∞ -algebra structure on V^* , the dual space to V .*

Proof. The first statement follows from the definition of an L_∞ coalgebra, while an L_∞ structure on V^* is a consequence of Theorem 17. \square

4.4. L_∞ -bialgebras. Following the general philosophy of structures given by solutions of Maurer-Cartan equations on differential graded Lie algebras, we consider solutions of an equation $[Q, Q] = 0$ where Q belongs to Lie subalgebras of $B : \mathcal{C}, \mathcal{L}, \mathcal{B}$ and \mathcal{QB} from Proposition 7. They respectively define an L_∞ -coalgebra, L_∞ -algebra, L_∞ -bialgebra and L_∞ -quasibialgebra structures on V . In all cases we consider the iterated adjoint action (10) of the corresponding Q on $\bigwedge V$.

Let us give two new definitions: for a differential Lie bialgebra structure and for a L_∞ bialgebra structure.

Definition 23. An L_∞ bialgebra structure on V is a set of maps

$$\tau_{kl} : \bigwedge^k V \rightarrow \bigwedge^l V[3 - (k + l)], \quad k, l \geq 1$$

of total degree 1 such that each $\tau_{kl} = \mathbf{iad}_{t_{kl}}$, $t_{kl} \in \bigwedge^k V^* \otimes \bigwedge^l V$ and

$$(12) \quad \sum_{k+k'=p+1} \sum_{l+l'=q+1} [t_{kl}, t_{k'l'}] = 0,$$

for all $p \geq 2, q \geq 2$.

In other words, an element of \mathcal{B} of total degree 1 :

$$(13) \quad T = \sum_{k \geq 1, l \geq 1} t_{kl}, \quad t_{kl} \in \bigwedge^k V^* \otimes \bigwedge^l V \text{ such that } [T, T] = 0.$$

defines an L_∞ -bialgebra structure on V .

We see that T lives in the following bi-graded space:

		...				
4	$\bigwedge^4 V^* \otimes V$...				
3	$\bigwedge^3 V^* \otimes V$	$\bigwedge^3 V^* \otimes \bigwedge^2 V$...			
2	$\bigwedge^2 V^* \otimes V$	$\bigwedge^2 V^* \otimes \bigwedge^2 V$	$\bigwedge^2 V^* \otimes \bigwedge^3 V$...		
1	$V^* \otimes V$	$V^* \otimes \bigwedge^2 V$	$V^* \otimes \bigwedge^3 V$	$V^* \otimes \bigwedge^4 V$...	
$k \quad / \quad l$	1	2	3	4		

Let us look at the first few equations from (12). The first one is

$$(14) \quad [t_{11}, t_{11}] = 0,$$

providing the equation which defines a differential $d = \mathbf{ad}_{t_{11}}$ on V . The equations for $p = 1, q = 2$ and $p = 2, q = 1$ give respectively:

$$(15) \quad [t_{11}, t_{12}] = 0, \quad [t_{11}, t_{21}] = 0,$$

which gives the condition that d be a derivation of the cobracket and of the bracket. The equation for $p = 2, q = 2$:

$$(16) \quad [t_{11}, t_{22}] + [t_{12}, t_{21}] = 0$$

shows that the bracket is a cocycle with respect to the cobracket up to homotopy. The homotopy is given by the element $t_{22} \in \bigwedge^2 V^* \otimes \bigwedge^2 V$.

The Jacobi identity for the bracket $\mathbf{iad}(t_{21})$ also holds only up to homotopy; it is given by the equation for $p = 3, q = 1$:

$$(17) \quad [t_{11}, t_{31}] + \frac{1}{2}[t_{21}, t_{21}] = 0.$$

The co-Jacobi identity up to homotopy is the equation for $p = 1, q = 3$:

$$(18) \quad [t_{11}, t_{13}] + \frac{1}{2}[t_{12}, t_{12}] = 0.$$

Remark 24. 1) *The homology of V , $H^*(V, d = \mathbf{ad} t_{11})$ is a Lie bialgebra.*

2) *If V is not graded, that is all $V_n = \{0\}$ for all $n \neq 0$, then $V = V_0$ is an ordinary Lie bialgebra. Indeed, since t_{kl} is of total degree 1, on an ungraded space only terms with k, l satisfying $1 - (k + l - 2) = 0$ survive. The result is that all t_{kl} except for t_{12} and t_{21} have to vanish. The Maurer-Cartan equation then gives the axioms of a Lie bialgebra from Definition 1: $\tau_{21} = \mathbf{iad} t_{21}$ defines a bracket and $\tau_{12} = \mathbf{iad} t_{12}$ a cobracket.*

3) *If all $t_{kl} = 0$ for all $k + l \geq 3$, then V is a differential graded Lie bialgebra (graded Lie bialgebra with a differential compatible with the bracket and the cobracket).*

4.5. L_∞ -quasi-bialgebra. To have an L_∞ -quasi-bialgebra structure we need to allow terms in $\bigwedge^q V$. Hence the Maurer-Cartan equation of the previous subsection (13) becomes an equation on $S = \sum_{k \geq 0, l \geq 1} t_{kl}$, $t_{kl} \in \bigwedge^k V^* \otimes \bigwedge^l V$ which differs from T because it contains elements $t_{0l} \in \bigwedge^l V$. This S must satisfy a set of equations indexed by $p \geq 1, q \geq 2$:

$$[S, S] = \sum_{k+k'=p} \sum_{l+l'=q} [t_{kl}, t_{k'l'}] = 0,$$

		...					
4	$\bigwedge^4 V^* \otimes V$...					
3	$\bigwedge^3 V^* \otimes V$	$\bigwedge^3 V^* \otimes \bigwedge^2 V$...				
2	$\bigwedge^2 V^* \otimes V$	$\bigwedge^2 V^* \otimes \bigwedge^2 V$	$\bigwedge^2 V^* \otimes \bigwedge^3 V$...			
1	$V^* \otimes V$	$V^* \otimes \bigwedge^2 V$	$V^* \otimes \bigwedge^3 V$	$V^* \otimes \bigwedge^4 V$...		
0			$\bigwedge^3 V$	$\bigwedge^4 V$...		
$(p+1) \quad \diagdown \quad (q+1)$		1	2	3	4		

Terms t_{0l} do not act on $\bigwedge V$, however they change the co-Jacobi condition and the other equations as well in comparison with the equations on T .

In particular, terms $t_{01} \in V$ and $t_{02} \in V \wedge V$ change the nature of certain equations: for example, $[t_{11}, t_{11}] = 0$ is no longer true in the presence of t_{01} . Allowing non-zero terms t_{01} and t_{02} would lead us to a completely different setup of weak L_∞ -algebras and to avoid that we impose that these terms are 0. Weak L_∞ -algebras appear in physics, vanishing of the terms t_{01} and t_{02} is related to certain boundary condition in Batalin-Vilkovisky formulation of the open string A-model (see [24]).

Definition 25. *Consider an element of \mathcal{QB} of total degree 1 :*

$$(19) \quad S = \sum_{k \geq 1, l \geq 1} t_{kl} + \sum_{l \geq 3} t_{0l}, \quad t_{kl} \in \bigwedge^k V^* \otimes \bigwedge^l V,$$

such that

$$(20) \quad [S, S] = 0.$$

An L_∞ quasi-bialgebra structure on V is the set of maps

$$\tau_{kl} : \bigwedge^k V \rightarrow \left(\bigwedge^l V \right) [3 - (k + l)], \quad k \geq 0, \quad l \geq 1$$

of total degree 1, where $\tau_{kl} = \mathbf{iad} \, t_{kl}$.

Let us state properties of an L_∞ -quasi-bialgebra similar to the properties stated in Remark 24:

Remark 26. 1) The homology of $H^*V = H^*(V, \mathbf{ad}t_{11})$ is a Lie quasi-bialgebra.

2) If V is not graded then V is an ordinary Lie quasibialgebra. Indeed, since t_{kl} is of degree $2 - (k + l - 1)$ all t_{kl} except for t_{12}, t_{21} and t_{03} have to vanish. Then $\mathbf{iad} \, t_{12}$ defines a cobracket, $\mathbf{iad} \, t_{21}$ a bracket and together with $t_{03} \in \bigwedge^3 V$ they satisfy the axioms of Lie quasibialgebras from Definition 2.

3) If all $t_{kl} = 0$ for all $k + l \geq 3$ then V is a differential graded Lie quasi-bialgebra.

Remark 27. For L_∞ -(quasi)bialgebras there is no analogue of Theorems 17, 22. For $Q \in B$ The condition $[Q, Q] = 0$ in general does not imply $(\mathbf{iad} \, Q)^2 = 0$. Although it is true for Q either in \mathcal{L} or in \mathcal{C} , in a more general case the equation $(\mathbf{iad} \, Q)^2 = 0$ force higher order conditions on Q which are not necessarily satisfied even for an ordinary (not L_∞) Lie bialgebra or a Lie quasi-bialgebra. For example for a Lie bialgebra the condition $(\mathbf{iad} \, Q)^2 = 0$ would force an extra axiom. Namely, the cobracket applied to an element and then the bracket to the result must be required to be 0, which gives a "no cycle" condition but it is not in the initial set of axioms for a Lie bialgebra. That kind of bialgebras "without cycles" appears in the work of Chas and Sullivan on string topology [5].

5. FORMAL GEOMETRY OF L_∞ -STRUCTURES.

We now move to a geometric definition of an L_∞ -algebra structure on a finite dimensional space (see for example [2] or [22]). Consider a finite dimensional graded vector space $V = \bigoplus_k V_k$ with a chosen basis $\{e_\alpha\}$. Then a Lie algebra structure is defined by structure constants $c_{\alpha,\beta}^\gamma$:

$$[e_\alpha, e_\beta] = \sum c_{\alpha,\beta}^\gamma e_\gamma,$$

satisfying the structure equation which boils down to the Jacobi identity. We could also see the bracket acting from $V \wedge V$ to V as a differential operator $Q_{21} = c_{\alpha,\beta}^\gamma e_\gamma \otimes \frac{\partial}{\partial e_\alpha} \wedge \frac{\partial}{\partial e_\beta}$. We use the index 21 for this operator to underline that it is quadratic in $\frac{\partial}{\partial e}$'s and linear in e 's.

Under the identification similar to (1)

$$(21) \quad \text{Sym}^n(X[1]) \simeq \left(\bigwedge^n(X) \right) [n]$$

a Lie bracket $\bigwedge^2 V \rightarrow V$ is a map of degree 1: if we shift the degree on the right and on the left by 2, $\left(\bigwedge^2 V \right) [2] \rightarrow V[2]$, we get equivalently $\text{Sym}^2(V[1]) \rightarrow (\text{Sym}(V[1]))[1]$. On the other hand, the dual of the free cocommutative coalgebra $\text{Sym}(V[1])$ could be identified with the algebra of formal power series on the \mathbb{Z} -graded space $V[1]$. We say that the algebra of formal power series defines a formal manifold. This way a Lie bracket becomes a degree 1 quadratic vector field on it. The dual space to $V[1]$ is $V^*[-1]$. Elements of $\text{Sym}(V^*[-1])$ are coordinate functions on $V[1]$. Let us chose corresponding coordinates $\{x^\alpha\}$, dual to e_α , such that in the space $V^*[-1]$ x^α has degree shifted by 1 : $\tilde{x}^\alpha + 1$.

A very natural generalization of this construction is the definition of an L_∞ -algebra structure:

Proposition 28. (see for example [11]) An L_∞ -algebra structure on a graded space V is given by vector field Q of degree 1 on the \mathbb{Z} -graded formal manifold corresponding to $V[1]$ such that $[Q, Q] = 0$.

Definition 29. (following Vaintrob [29]) A non-zero self-commuting operator (and in particular a vector field) on a formal manifold is called homological.

In particular an L_∞ -algebra structure is given by a homological vector field:

$$Q = b_\alpha^\beta x^\alpha \frac{\partial}{\partial x^\beta} + c_{\alpha\beta}^\gamma x^\alpha x^\beta \frac{\partial}{\partial x^\gamma} + d_{\alpha\beta\gamma}^\delta x^\alpha x^\beta x^\gamma \frac{\partial}{\partial x^\delta} + \dots,$$

$$Q = \sum_{k=1}^{\infty} Q_{k1}, \quad Q_{k1} = f_{\alpha_1 \dots \alpha_k}^\beta x^{\alpha_1} \dots x^{\alpha_k} \frac{\partial}{\partial x^\beta}.$$

In fact, each term in Q corresponds to a map of degree 1 on the dual space:

$$(22) \quad Q = \sum Q_{k1}, \quad Q_{k1} : V^*[-1] \rightarrow (Sym^k(V^*[-1]))[1].$$

Counting degrees after the identification (21) we get $Q_{k1} : V^*[-1] \rightarrow (\bigwedge^k V^*[-k])[1]$ which on the dual space becomes a map $\bigwedge^k V \rightarrow V[2-k]$ as in Definition 15.

This way we are naturally brought to a geometric definition of an L_∞ -bialgebra structure on V . In the sum (22) we should consider not only $Q_{k1} : V^* \rightarrow Sym^k V^*$, but any $Q_{km} : Sym^m V^* \rightarrow Sym^k V^*$. It becomes a differential operator on $V[1]$, acting on functions on $V[1]$, which are elements of $Sym(V^*[-1])$.

According to Definition 12 the operator Q_{km} has the external degree $\mathbf{ed}(Q_{km}) = k + m - 2$, following the external grading in the space $\Lambda^m V^* \otimes \Lambda^k V$. The total degree of Q_{km} is then a difference of internal degrees of the result with the initial element plus the external degree $k + m - 2$.

We consider the ring of differential operators on $V[1]$:

$$\mathbb{A}^h(V) := Sym(V^*[-1])[\xi^1, \dots, \xi^n]$$

where $\xi^\alpha = \frac{\partial}{\partial x_\alpha}$. Notice that $\xi^\alpha \in (V[1])$. On $\mathbb{A}^h(V)$ there is a standard graded Poisson bracket of [4, 9]. For any two elements $E, F \in \mathbb{A}^h(V)$ it could be defined in coordinates as follows:

$$\{E, F\} = \frac{\partial E}{\partial x^\alpha} \frac{\partial F}{\partial \xi_\alpha} - (-1)^{\tilde{E}\tilde{F}} \frac{\partial F}{\partial x^\alpha} \frac{\partial E}{\partial \xi_\alpha}$$

Proposition 30. Any L_∞ -bialgebra structure on V is given by an operator of degree 1 on the corresponding formal manifold. The Poisson bracket of this operator with itself is 0.

Proof. Consider an operator $Q = \sum Q_{pq}$, where $Q_{pq} = v^1 \dots v^p w_1^* \dots w_q^*$. Being of degree 1 gives the following equality: $\sum(\tilde{v}_i + \tilde{w}_j) + p + q - 2 = 1$. Here $\tilde{w}_j = -\tilde{w}_j^*$. The operator Q acts on skew-symmetric algebraic functions on V , that is on $Sym(V^*[-1])$. Thus, after the identification (1) we get that $Q_{pq} : \bigwedge^p V^* \rightarrow (\bigwedge^q V^*)[1][2-(p+q)]$ which on the dual space gives $\bigwedge^q V[p+q-3] \rightarrow \bigwedge^p V$ leading to the right degrees.

The Poisson bracket of Q with itself being 0 corresponds to the self-commuting condition in (13). \square

6. MANIN L_∞ -TRIPLE AND L_∞ -PAIR

We give a definition of a Manin L_∞ -triple (which also could be called a strongly homotopy Manin triple), a natural generalization of a Manin triple. For this we need a notion of an L_∞ -subalgebra:

Definition 31. Let (W, Q) be an L_∞ -algebra. Then a subspace $V \subset W$ is an L_∞ -subalgebra if V is Q -invariant.

It means that V is an L_∞ -subalgebra of W if the image of the operator Q acting on $V^*[-1]$ is in $Sym V^*[-1][1]$. In other words it means that $Q = \sum_{k,l \geq 1} Q_{kl}$ restricted to V consists of maps $V^* \rightarrow \bigwedge^k V^*[2-k]$ or, on the dual, it gives an L_∞ structure on V , $Q_{kl} : \bigwedge^k V \rightarrow V[2-k]$.

Definition 32. A finite dimensional Manin L_∞ -triple is a triple of finite dimensional L_∞ -algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ equipped with a nondegenerate bilinear form \langle, \rangle such that

- $\mathfrak{g}_+, \mathfrak{g}_-$ are L_∞ -subalgebras of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as a vector space;
- \mathfrak{g}_+ and \mathfrak{g}_- are isotropic with respect to \langle, \rangle ;
- the n -brackets constituting the L_∞ -algebra structure $\lambda_n : \bigwedge^n (V^* \oplus V) \rightarrow V^* \oplus V$ are invariant with respect to the bilinear form \langle, \rangle , that is

$$(23) \quad \langle \lambda_n(v_1 \wedge \cdots \wedge v_n), v_0 \rangle = (-1)^{\widetilde{v}_n \widetilde{v}_0} \langle \lambda_n(v_1 \wedge \cdots \wedge v_{n-1} \wedge v_0), v_n \rangle$$

Notice that a Manin triple is an example of a Manin L_∞ -triple.

The invariance is in fact cyclic, since λ_n (being a map on $\bigwedge^n V$) is antisymmetric in all variables.

Theorem 33. The notions of a (finite dimensional) Manin L_∞ -triple and an L_∞ -bialgebra are equivalent.

Proof. Consider a Manin L_∞ -triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$. Let us show that there is an L_∞ -bialgebra structure on \mathfrak{g}_+ . Let the L_∞ -algebra structure on \mathfrak{g} be given by maps $\lambda_k : \bigwedge^k \mathfrak{g} \rightarrow \mathfrak{g}$, that is $\lambda_k : \bigwedge^k (\mathfrak{g}_+ \oplus \mathfrak{g}_-) \rightarrow (\mathfrak{g}_+ \oplus \mathfrak{g}_-)$. These maps can be split as follows:

$$\lambda_{mn+} : \bigwedge^m \mathfrak{g}_+ \otimes \bigwedge^n \mathfrak{g}_- \rightarrow \mathfrak{g}_+ \text{ and } \lambda_{mn-} : \bigwedge^m \mathfrak{g}_+ \otimes \bigwedge^n \mathfrak{g}_- \rightarrow \mathfrak{g}_- \text{ with } m+n=k.$$

Using non-degeneracy and invariance (23) of the internal product \langle, \rangle after an identification of \mathfrak{g}_-^* with \mathfrak{g}_+ , we get the equivalences

$$\begin{aligned} \text{Hom}_{inv}(\bigwedge^m \mathfrak{g}_+ \otimes \bigwedge^n \mathfrak{g}_-, \mathfrak{g}_+) &\simeq \text{Hom}(\bigwedge^m \mathfrak{g}_+, \bigwedge^{n+1} \mathfrak{g}_+) \simeq \bigwedge^m \mathfrak{g}_+^* \otimes \bigwedge^{n+1} \mathfrak{g}_+ \text{ and} \\ \text{Hom}_{inv}(\bigwedge^m \mathfrak{g}_+ \otimes \bigwedge^n \mathfrak{g}_-, \mathfrak{g}_-) &\simeq \text{Hom}(\bigwedge^{m+1} \mathfrak{g}_+, \bigwedge^n \mathfrak{g}_+) \simeq \bigwedge^{m+1} \mathfrak{g}_+^* \otimes \bigwedge^n \mathfrak{g}_+. \end{aligned}$$

Let $l_{mn+} \in \bigwedge^m \mathfrak{g}_+^* \otimes \bigwedge^{n+1} \mathfrak{g}_+$ correspond to λ_{mn+} . This way, l_{mn+} would act by a iterated adjoint action:

$$\mathbf{iad}(l_{mn+}) : \bigwedge^m \mathfrak{g}_+ \rightarrow \bigwedge^{n+1} \mathfrak{g}_+.$$

On the other hand $\lambda_{mn-} : \bigwedge^m \mathfrak{g}_+ \otimes \bigwedge^n \mathfrak{g}_- \rightarrow \mathfrak{g}_-$ again is given by a iterated adjoint action of an element $l_{mn-} \in \bigwedge^{m+1} \mathfrak{g}_+^* \otimes \bigwedge^n \mathfrak{g}_+$ and

$$\mathbf{iad}(l_{mn-}) : \bigwedge^{m+1} \mathfrak{g}_+ \rightarrow \bigwedge^n \mathfrak{g}_+.$$

Now, $L = \sum_{m,n \geq 1} (l_{mn+} + l_{mn-})$ satisfies $[L, L] = 0$ from the condition on λ .

In the other direction it is actually easier. Let an L_∞ -bialgebra structure on V be defined by the iterated adjoint action of

$$T = \sum_{k \geq 1, l \geq 1} t_{kl}, \text{ where } t_{kl} \in \bigwedge^k V^* \otimes \bigwedge^l V.$$

Then the corresponding Manin L_∞ -triple is $(V \oplus V^*, V, V^*)$ with the nondegenerate form given by the natural pairing of V and V^* .

The L_∞ structure on $V^* \oplus V$ is given by $\tau = \mathbf{iad}(T)$. In

$$\mathbf{iad}(t_{kl}) : (V^* \oplus V) \rightarrow \bigwedge^{k-1} V^* \otimes \bigwedge^l V \oplus \bigwedge^k V^* \otimes \bigwedge^{l-1} V$$

we recognize the L_∞ -coalgebra structure dual to an L_∞ -algebra structure we are looking for.

To get the L_∞ -subalgebra structure on V we consider the quadratic equations (12) for $q = 2$. We get:

$$\sum_{k+k'=p} t_{k1} t_{k'1} = 0,$$

the set of equations defining an L_∞ -structure on V (by taking $t_{k1} = l_k$ in (17)).

The operator $\tau_V = \sum_{k \geq 1} \mathbf{iad}(t_{k1}) : V^* \rightarrow \bigwedge V^*$ is the restriction of τ to V^* making V an L_∞ -subalgebra of $V^* \oplus V$, since T_V is self-commuting.

In the same way, elements $t_{1l} \in V^* \otimes \bigwedge^l V$ satisfy the following quadratic equation (equations (12) for $p = 2$):

$$\sum_{l+l'=q} t_{1l} t_{1l'} = 0$$

therefore they induce maps $\mathbf{iad}(t_{1l}) : V \rightarrow \bigwedge^l V$, on V thus defining an L_∞ -algebra structure on V^* . \square

The notion which leads to an L_∞ -quasi-bialgebra is a Manin L_∞ -pair (strongly homotopy Manin pair):

Definition 34. *A finite dimensional Manin L_∞ -pair is a pair of finite dimensional L_∞ -algebras $\mathfrak{g} \supset \mathfrak{g}_+$ equipped with a nondegenerate invariant (23) bilinear form \langle, \rangle , such that \mathfrak{g}_+ is an isotropic L_∞ -subalgebra of \mathfrak{g} .*

A Manin L_∞ -quasi-triple is a Manin L_∞ -pair $(\mathfrak{g}, \mathfrak{g}_+)$ with a chosen Lagrangian complement of \mathfrak{g}_+ .

There is also a correspondence as in Theorem 33 with a similar proof:

Theorem 35. *The notions of a (finite dimensional) Manin L_∞ -quasi-triple and an L_∞ -quasi-bialgebra are equivalent.*

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UNIVERSITÉ DE LYON, UNIVERSITÉ LYON1, CNRS, UMR 5208 INSTITUT CAMILLE JORDAN, BATIMENT DU DOYEN
JEAN BRACONNIER, 43, BLVD DU 11 NOVEMBRE 1918, F - 69622 VILLEURBANNE CEDEX, FRANCE

E-mail address: ok@alum.mit.edu